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# **A COMPARISON OF TWO TECHNIQUES FOR THE INTERPOLATION/ EXTRAPOLATION OF FREQUENCY DOMAIN RESPONSES**

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# **A COMPARISON OF TWO TECHNIQUES FOR THE INTERPOLATION/EXTRAPOLATION OF FREQUENCY DOMAIN RESPONSES**

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**ABSTRACT:** In this paper, a comparison is made between two methods which are used for interpolation/extrapolation of frequency domain responses.

The first is the direct method based on the principle of a model based parameter estimation and the second method uses the properties of discrete Hilbert transforms based on the principle of causality.

## **1. INTRODUCTION**

In a host of applications in engineering, it is necessary to obtain information about a system over a broad range. In most cases, it is not possible to evaluate the parameter of interest in a closed form. However, either theoretical or experimental data is available in a narrow band. The first method uses the principle of analytic continuation to extrapolate/interpolate the data over a wide band. The second method uses the property of Hilbert transform which relates the real and imaginary components of the frequency domain.

In Section 2, the direct method is studied. Section 3 describes the numerical implementation of the Hilbert transform. Section 4 describes the algorithm used to implement this. In Section 5, few numerical examples are studied and comparisons made.

## 2. THE CAUCHY METHOD

Consider a system function  $H(s)$ . The objective is to approximate  $H(s)$  by a ratio of two polynomials  $A(s)$  and  $B(s)$  so that  $H(s)$  can be represented by fewer variables.

Hence, consider

$$H(s) \approx \frac{A(s)}{B(s)} = \frac{\sum_{k=0}^P a_k s^k}{\sum_{k=0}^Q b_k s^k} \quad (1)$$

Here, the given information could be the value of the function  $H(s)$  and its  $N_j$  derivatives at some frequency points  $s_j, j=1, \dots, J$ . If  $H^n(s_j)$  represents the  $n^{\text{th}}$  derivative of  $H(s)$  at point  $s=s_j$ , the Cauchy problem is:

Given  $H^{(n)}(s_j)$  for  $n=0, \dots, N_j, j=1, \dots, J$ , find  $P, Q, \{a_k, k=0, \dots, P\}$ , and  $\{b_k, k=0, \dots, Q\}$ .

The solution for  $\{a_k\}$  and  $\{b_k\}$  is unique if the total number of samples is greater than or equal to the total number of unknown coefficients  $P - Q + 2$  [1], i.e.

$$N \equiv \sum_{j=1}^J (N_j + 1) \geq P + Q + 2$$

By enforcing the equality in equation (1) one obtains

$$A(s) = H(s)B(s) \quad (2)$$

Differentiating the above equation  $n$  times, and evaluating the expressions at point  $s_j$ , results in the binomial expansion,

$$A^{(n)}(s_j) = \sum_{i=0}^n {}^nC_i H^{(n-i)}(s_j) B^{(i)}(s_j) \quad (3)$$

where,

$${}^nC_i = \frac{n!}{(n-i)!i!},$$

$n!$  represents the factorial of  $n$ .

Using the polynomial expansions for  $A(s)$  and  $B(s)$ , equation (3) can be written as

$$\sum_{k=0}^P A_{(j,n),k} s_k^* = \sum_{k=0}^Q B_{(j,n),k} b_k \quad (4)$$

where,

$$A_{(j,n),k} = \frac{k!}{(k-n)!} s_j^{(k-n)} u(k-n) \quad (5)$$

$$B_{(j,n),k} = \sum_{i=0}^n C_i H^{(n-i)}(s_j) u(k-i) \frac{k!}{(k-i)!} s_j^{(k-i)} \quad (6)$$

$n=0, 1, \dots, N_j$ ,  $j=1, \dots, J$ , where  $u(k)=0$  for  $k<0$  and  $=1$  otherwise.

Define,

$$\mathbf{A} = [A_{(j,n),0}, A_{(j,n),1}, \dots, A_{(j,n),P}] \quad (7)$$

$$\mathbf{B} = [B_{(j,n),0}, B_{(j,n),1}, \dots, B_{(j,n),Q}] \quad (8)$$

$$[\mathbf{a}] = [a_0, a_1, a_2, \dots, a_P]^T \quad (9)$$

$$[\mathbf{b}] = [b_0, b_1, b_2, \dots, b_Q]^T \quad (10)$$

The order of matrix  $\mathbf{A}$  is  $N \times (P + 1)$  and that of  $\mathbf{B}$  is  $N \times (Q + 1)$ . Then, equation (4) becomes

$$[\mathbf{A}]\mathbf{a} = [\mathbf{B}]\mathbf{b} \quad (11)$$

or

$$[\mathbf{A} \mid -\mathbf{B}] \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \mathbf{0} \quad (12)$$

For ease of notation, define  $[\mathbf{C}] \equiv [\mathbf{A} \mid -\mathbf{B}]$ .  $\mathbf{C}$  is of order  $N \times (P + Q + 2)$ . A Singular Value Decomposition (SVD) of the matrix  $\mathbf{C}$  will give us a gauge of the required values of  $P$  and  $Q$  [2]. A SVD results in the equation

$$[U][\Sigma][V]^H \begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad (13)$$

The matrices  $U$  and  $V$  are unitary matrices and  $\Sigma$  is a diagonal matrix with the singular values of  $C$  in descending order as its entries. The columns of  $U$  are the left eigenvectors of  $B$  or the eigenvectors of  $CC^H$ . The columns of  $V$  are the right eigenvectors of  $C$  or the eigenvectors of  $C^H C$ . The singular values are the square roots of the eigenvalues of the matrix  $C^H C$ . Therefore, the singular values of any matrix are real and positive. The number of nonzero singular values is the rank of the matrix in equation (12) and so gives us an idea of the information in this system of simultaneous equations. If  $R$  is the number of nonzero singular values, the dimension of the right null space of  $C$  is  $P + Q + 2 - R$ . Our solution vector belongs to this null space. Hence to make this solution unique, we need to make the dimension of this null space 1 so that only one vector defines this space. Hence  $P$  and  $Q$  must satisfy the relation

$$R + 1 = P + Q + 2 \quad (14)$$

The solution algorithm must include a method to estimate  $R$ . This is done by starting out with the choices of  $P$  and  $Q$  that are higher in dimension than can be expected for the system at hand. Then we get an estimate for  $R$  from the number of non-zero singular values of the matrix  $C$ . Now, using equation (14) we get better estimates for  $P$  and  $Q$ . Letting  $P$  and  $Q$  stand for these new estimates of the polynomial orders, we can recalculate the matrices  $A$  and  $B$ . Therefore, we come back to the relation

$$[C] \begin{bmatrix} a \\ b \end{bmatrix} \equiv [A \mid -B] \begin{bmatrix} a \\ b \end{bmatrix} = 0 \quad (15)$$

$[C]$  is a rectangular matrix with more rows than columns. Many methods to solve equation (15) are well documented [2]. For reasons indicated in the appendix, we choose the method of Total Least Squares (TLS) [3].

### 3. TRANSFORM RELATIONSHIPS

This section briefly covers some of the properties of sequences and their Fourier

transforms. Any complex sequence  $h[n]$  can be expressed as a sum of a symmetric sequence  $h_e[n]$  and an anti-symmetric sequence  $h_o[n]$ . In the case of real sequences, these are called even and odd sequences [4]. Therefore,

$$h[n] = h_e[n] + h_o[n] \quad (16)$$

$$h_e[n] = h_e[-n] \quad (17)$$

$$h_o[n] = -h_o[-n] \quad (18)$$

The Fourier transform of any complex sequence  $h[n]$  is represented by  $H(e^{j\omega})$ , where

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \quad (19)$$

Therefore,

$$H(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{j\omega n} \quad (20)$$

This implies,

$$H_R(e^{j\omega}) = \mathcal{R}\{h_e[n]\} \quad (21)$$

and

$$jH_S(e^{j\omega}) = \mathcal{I}\{h_o[n]\} \quad (22)$$

Also, for real  $h[n]$ ,  $H_R(e^{j\omega}) = H_R(e^{-j\omega})$  which is an even function, and  $H_S(e^{j\omega}) = -H_S(e^{-j\omega})$  which is an odd function.

### 3.1. THE HILBERT TRANSFORM RELATIONSHIP FOR A CAUSAL SEQUENCE

Consider a periodic, real, time domain sequence  $h_p[n]$  with period  $N$ , that is related to a finite length sequence  $h[n]$  of length  $N$  by

$$h_p[n] = \sum_{i=-\infty}^{\infty} h[n+iN] \quad (23)$$

From Eq. (16) we have,

$$h_p[n] = h_{pe}[n] + h_{po}[n] \quad (24)$$

and from Eq. (17) we have

$$h_{pe}[n] = \frac{1}{2} \{h_p[n] + h_p[-n]\} \quad (25)$$

and similarly from Eq. (18),

$$h_{po}[n] = \frac{1}{2} \{h_p[n] - h_p[-n]\} \quad (26)$$

If we have  $N=2r$  (where  $r$  is a positive integer) then,

$$h_p[n] = \begin{cases} 2h_{pe}[n] & n = 1, \dots, \frac{N}{2}-1 \\ h_{pe}[n] & n = 0, \frac{N}{2} \\ 0 & n = \frac{N}{2} + 1, \dots, N-1 \end{cases} \quad (27)$$

Also the odd part of the sequence can be expressed as,

$$h_{po}[n] = \begin{cases} h_{pe}[n] & n = 1, \dots, \frac{N}{2}-1 \\ 0 & n = 0, \frac{N}{2} \\ -h_{pe}[n] & n = \frac{N}{2} + 1, \dots, N-1 \end{cases} \quad (28)$$

If we define  $u_{pN}[n]$  as the periodic sequence

$$u_{pN}[n] = \begin{cases} 2 & n = 1, \dots, \frac{N}{2}-1 \\ 1 & n = 0, \frac{N}{2} \\ 0 & n = \frac{N}{2} + 1, \dots, N-1 \end{cases} \quad (29)$$

we can express  $h_p[n]$  as

$$h_p[n] = h_{pe}[n]u_{pN}[n] \quad (30)$$

Equivalently, the Fourier transform of  $h_p[n]$  yields,

$$H_p[k] = \frac{1}{N} \sum_{m=0}^{N-1} H_{p\Re}[m]U_{pN}[k-m] = H_{p\Re}[k] + jH_{p\Im}[k] \quad (31)$$

Eq. (29) can be alternatively expressed as,

$$u_{pN}[n] = 2u[n] - 2u[n - \frac{N}{2}] - \delta[n] + \delta[n - \frac{N}{2}] \quad (32)$$

where  $u[n]$  is the unit step sequence, and  $\delta[n]$  is the unit sample sequence [10].

The Fourier transform of  $u_{pN}[n]$  is,

$$U_{pN}[k] = \begin{cases} -2j\cot\left[\frac{\pi k}{N}\right] & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \quad (33)$$

with  $U_{pN}[0] = N$  as derived from the definition of  $U_{pN}[k]$ .

Defining

$$V_{pN}(k) = U_{pN}(k) - N\delta(k) \quad (34)$$

we have, from Eq. (31)

$$H_p[k] = H_{p\Re}[k] + \frac{1}{N} \sum_{m=0, m \neq k}^{N-1} H_{p\Re}[m] V_{pN}[k-m] \quad (35)$$

Hence we get the imaginary part of the Fourier transform as,

$$jH_{p\Im}[k] = \frac{1}{N} \sum_{m=0, m \neq k}^{N-1} H_{p\Re}[m] V_{pN}[k-m] \quad (36)$$

This is the Hilbert transform relationship between the real and imaginary parts of the Fourier transform of a periodically causal sequence. If  $h[n] = 0$  for  $n < 0$  and for  $n > N/2$  then the periodicity maybe removed and we have the same relationships between the real and imaginary parts of the Fourier transform of  $h[n]$ .

$$jH_{\Im}[k] = \begin{cases} \frac{1}{N} \sum_{m=0, m \neq k}^{N-1} H_{\Re}[m] V_N[k-m] & 0 \leq k \leq N-1 \\ 0 & \text{otherwise} \end{cases} \quad (37)$$

In the previous expression the cotangent term is itself periodic with period  $N$ , so when we compute the  $N$  point discrete Fourier transform of the real causal time sequence, the relationship between the real and imaginary parts will be affected. Eq. (37) is the Hilbert transform relationship between  $H_{\Re}[k]$  and  $H_{\Im}[k]$ . Thus with a knowledge of one the other can be evaluated. Alternatively, since  $h_e[n]$  is the inverse Fourier transform of  $H_{\Re}[k]$ , it can be obtained by an Inverse Discrete Fourier transform (IDFT) of  $H_{\Re}[k]$ . Eq. (28) expresses a relationship between  $h_e[n]$  and  $h_o[n]$ , thus  $h_o[n]$  is known.  $H_{\Im}[k]$  can be obtained as the Fourier transform  $h_o[n]$ , by utilizing a discrete Fourier transform (DFT) algorithm. This procedure forms the basis of our technique for the extraction of a real, causal time domain response from band-limited complex frequency domain data. The theoretical development assures us that by computing the DFT's and IDFT's the original real time sequence will not lose its causal nature.

#### 4. INTERPOLATION/EXTRAPOLATION OF FREQUENCY DOMAIN DATA

A technique to extrapolate/interpolate data in the frequency domain utilizing the Hilbert Transform is described next. Before the algorithm is described, it is useful to know something

about the available frequency domain data. Assume that we have a complex frequency domain data between frequencies  $f_1$  and  $f_4$ . Consider a missing band between  $f_2$  and  $f_3$ . The frequency domain data is sampled at  $(n_2-n_1)$  frequency points between  $f_2$  and  $f_1$ , and at  $(n_4-n_3)$  points between  $f_4$  and  $f_3$ . This is expressed as a vector

$$H[n_1:n_4] = [H_{n1} \dots H_{n2}, 0 \dots 0, H_{n3} \dots H_{n4}] \quad (38)$$

It is now our objective to interpolate this missing data between  $n_{2+1}$  and  $n_{3-1}$ .

1. The available bandlimited frequency domain data is padded with zeros to ensure a length of  $n$  points where  $n$  is given by  $N/2 + 1$ , and  $N$  is  $[2, 4, 8, \dots, 1024, 2048, \dots]$ , providing a sequence of even length. The complex data is now given by

$$H[1:n] = H[1:N/2 + 1] = [0, 0, \dots, H_{n1}, \dots, H_{n2}, 0, 0, \dots, 0, H_{n3}, \dots, H_{n4}, 0, 0, \dots, 0] \quad (39)$$

2. This complex sequence is copied to obtain a complex consequence of length  $N$ . This is done by flipping the complex conjugate of the sequence and appending to the given sequence.

$$H[1:N] = [H[1:N/2 + 1], H^*[N/2:2]] \quad (40)$$

3. The complex sequence is now split into its real and imaginary parts.

$$H_R = \text{Real} [H] \quad (41)$$

$$H_I = \text{Imag} [H] \quad (42)$$

4. An inverse discrete Fourier transform of  $H_R$  results in an even sequence as stated earlier.

$$h_e(1:N) = \text{Real}[IFFT(H_R)] \quad (43)$$

This is in fact the even part of the actual time domain sequence.

5. Before proceeding further, it is important to know that there are sharp discontinuities in the frequency domain signal. In order to deal with this situation,

we will have to multiply the time domain sequence with a window.

The Hanning window of length  $N$  is multiplied with the time domain sequence. The resulting frequency domain sequence will now be "smeared" or smoothed out [10]. The Hanning window is given by

$$W(n) = \begin{cases} 0.5 - 0.5 \cos(2\pi n)/N & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \quad (44)$$

Hence,

$$h_e(1:N) = h_o(1:N) * W(1:N) \quad (45)$$

6. The odd sequence is obtained from the even sequence by making use of Eq. (28).  
We have

$$h_o(1:N) = [0 \quad h_e(2:N/2) \quad 0 \quad -h_e(N/2+2:N)] \quad (46)$$

7. The discrete Fourier transform of this odd sequence will give the imaginary part of the spectrum as stated earlier.

$$H_I^{\text{new}} = \text{Imag}[\text{FFT}(h_o)] \quad (47)$$

8. A substitution for the missing points is made in the imaginary part of the original sequence using the sequence obtained in step (7).

$$H_I^{\text{Sub}} = [H_I^{\text{new}}(1:n_1-1), H_I(n_1:n_2), H_I^{\text{new}}(n_2+1:n_3-1), H_I(n_3:n_4), H_I^{\text{new}}(n_4+1:N/2+1)] \quad (48)$$

9. This sequence is copied to obtain a sequence of length  $N$ .

$$H_I^{\text{Subs}} = [H_I^{\text{Sub}}[1:N/2+1], -H_I^{\text{Sub}}[N/2+2:N]] \quad (49)$$

which is an improved version of the original sequence  $H_I$ .

10. The inverse discrete Fourier transform of this sequence will give us the odd sequence again.

$$h_o^{new} = \text{IFFT}[j * H_I^{Subs}] \quad (50)$$

11. Again using Eqn. (9), we get the modified version of  $h_o$ .

$$h_o^{new} = [h_o(1), h_o^{new}(2:n/2), h_o(N/2+1), -h_o^{new}(N/2+2:N)] \quad (51)$$

12. The discrete Fourier transform of this sequence obtained in the previous step will give us the Real part of the spectrum as stated earlier.

$$H_R^{new} = \text{Real}[\text{FFT}(h_o^{new})] \quad (52)$$

13. A substitution for the missing points is made in the Real part of the original sequence using the sequence obtained in step (12).

$$H_R^{Sub} [H_R^{new}(1:n_1-1), H_R(n_1:n_2), H_R^{new}(n_2+1:n_3-1), H_R(n_3:n_4), H_R^{new}(n_4+1:N/2+1)] \quad (53)$$

14. This sequence is copied to obtain a sequence of length N.

$$H_R^{Subs} = [H_R^{Sub}[1:N/2+1], H_R^{Sub}[N/2+2:N]] \quad (54)$$

which is an improved version of the original sequence  $H_R$ .

15. The resulting sequence is subject to an inverse discrete Fourier transform to obtain the even sequence.

$$h_o(1:N) = \text{Real}[\text{IFFT}(H_R^{Subs})] \quad (55)$$

16. As in step 5, this time domain sequence is multiplied with the Hanning window.  
17. Subsequent signal processing are iterations of steps 6 through 16.

The above procedure will interpolate the missing band of frequencies. The reconstructed sequence will now be the complex sequence given by

$$H^{Rec}[1:n_4] = H_R^{Subs}[1:n_4] + jH_I^{Subs}[1:n_4] \quad (56)$$

comparing with Eqn. (38)

$$H^{Rec}[1:n_4] = [H_{n_1} \dots, H_{n_2}, \dots, H_{n_2+1}^{Rec}, \dots, H_{n_3-1}^{Rec}, H_{n_3} \dots H_{n_4}] \quad (57)$$

It is worthwhile to note that by making use of the Hanning window, although we have overcome the difficulties due to discontinuities at the ends of the missing band, we might suffer a loss of resolution. This is not a serious problem and its effects can be minimized as shown in the numerical examples.

Subsequent sections illustrate this technique.

## 5. NUMERICAL RESULTS

Let us first consider the frequency domain data of a microstrip filter measured using the HP 8510B Network Analyzer. The device is a bandpass filter and its characteristics are measured at 415 points from 4.31 GHz to 7.415GHz. Our objective now is to compare the performance of the two methods (The Cauchy method and the iterative technique based on the Hilbert transform) as the number of missing points are gradually increased. These missing points are created by deleting portions of the measured data.

Figure 1a shows the real and imaginary parts of the original data. Let us now discard 40 points (which is about 10% of the data) from 200-240. Figure 1b shows this deleted band of data. This data is now given as input to the direct method. The entire data set is not required for this method. Only a few points before and after the missing band is sufficient. The program returns the interpolated data. Next, this data is given as input to the iterative method utilizing the iterative technique based on the Hilbert transform presented in Section 4. The missing points are zero padded. Figure 1c compares the output of both methods with the original real part, while Figure 1d compares the corresponding imaginary part. Clearly the reconstruction is quite accurate using either techniques.

Next, the number of deleted points was increased to 60, i.e., points 200-260 were discarded. The same procedure was repeated. Figure 2a shows the truncated data. Figure 2b shows the reconstructed real part and Figure 2c shows the reconstructed imaginary part of the response utilizing both the techniques. It is clear from these figures that the reconstruction obtained using the iterative method based on Hilbert transform is slightly better than that obtained

using the direct method. Note that the amplitude of the reconstructed part using the Cauchy method came out slightly higher than the actual amplitude.

When the number of deleted points was increased to 80 [about 20% of the data has been deleted from the middle of the band], i.e., from 200-280, the iterative method again proved to be better than the direct method. But this time, a slight modification was made in the initial guess for the missing points in the iterative method. A straight line extrapolation between the ends of the missing band was made in the initial guess instead of zero-padding it. Figure 3a shows the original data with the initial guess. Figures 3b and 3c are the reconstructed real and imaginary parts using both methods. Clearly the iterative method gave better results. But it should be noted that a better interpolation can be got by the Cauchy method if the cut-off for the singular values, (explained earlier in the theory) is chosen appropriately. Figures 3d and 3e show the improved result. The singular value cut-off was changed to  $1.e^{21}$  from the previous value of  $1.e^{-16}$ . However, determining the cut-off for the singular values in practical situations may not be possible!

As the next example, we consider the frequency domain data of another microstrip bandpass filter measured at 468 points from 4.206875 GHz to 8.00125 GHz. This data was incomplete. That is, the entire data set was not considered as can be seen in Figure 4a, where the response in the stop band is not near zero. Due to this reason, the initial guess had to be made as the straight line extrapolation explained earlier.

Figure 4a shows the real and imaginary parts of the original data. Again we discard 40 points from 200-240 as shown in Figure 4b. Since the missing band was quite small, interpolation results using both techniques were similar. Figure 4c shows the reconstructed real parts compared with the original real part and Figure 4d shows the corresponding imaginary part.

Next, 60 points were discarded from 200-260. Now, knowing that the data itself is incomplete (explained earlier) and the missing band is large, modification in the initial guess was made in the iterative method. The initial guess was again the straight line extrapolation between the ends of the missing band. Figure 5a shows the original data with the initial guess. Figures 5b and 5c now show the reconstructed real and imaginary parts using both techniques. The results came out to be quite similar. Note that modifications in the iterative method can be made just by observing the given data and assessing the amount of information provided.

Finally, 80 points from 200-280 were considered missing. This meant that the interpolation was to be carried out for a very large missing band considering that the data itself was incomplete. Figure 6a shows the missing band and the straight line initial guess for the iterative method.

Figures 6b and 6c show the reconstructed real and imaginary parts compared with the original real and imaginary parts. Note that the Cauchy method gave better results. The limitation for the iterative technique based on the Hilbert transform was that the given data itself was incomplete to get better results. So if the data set is incomplete and the missing band data is to be interpolated/extrapolated then the Cauchy method provides slightly better performance than the iterative method based on the Hilbert Transform.

## 6. CONCLUSION

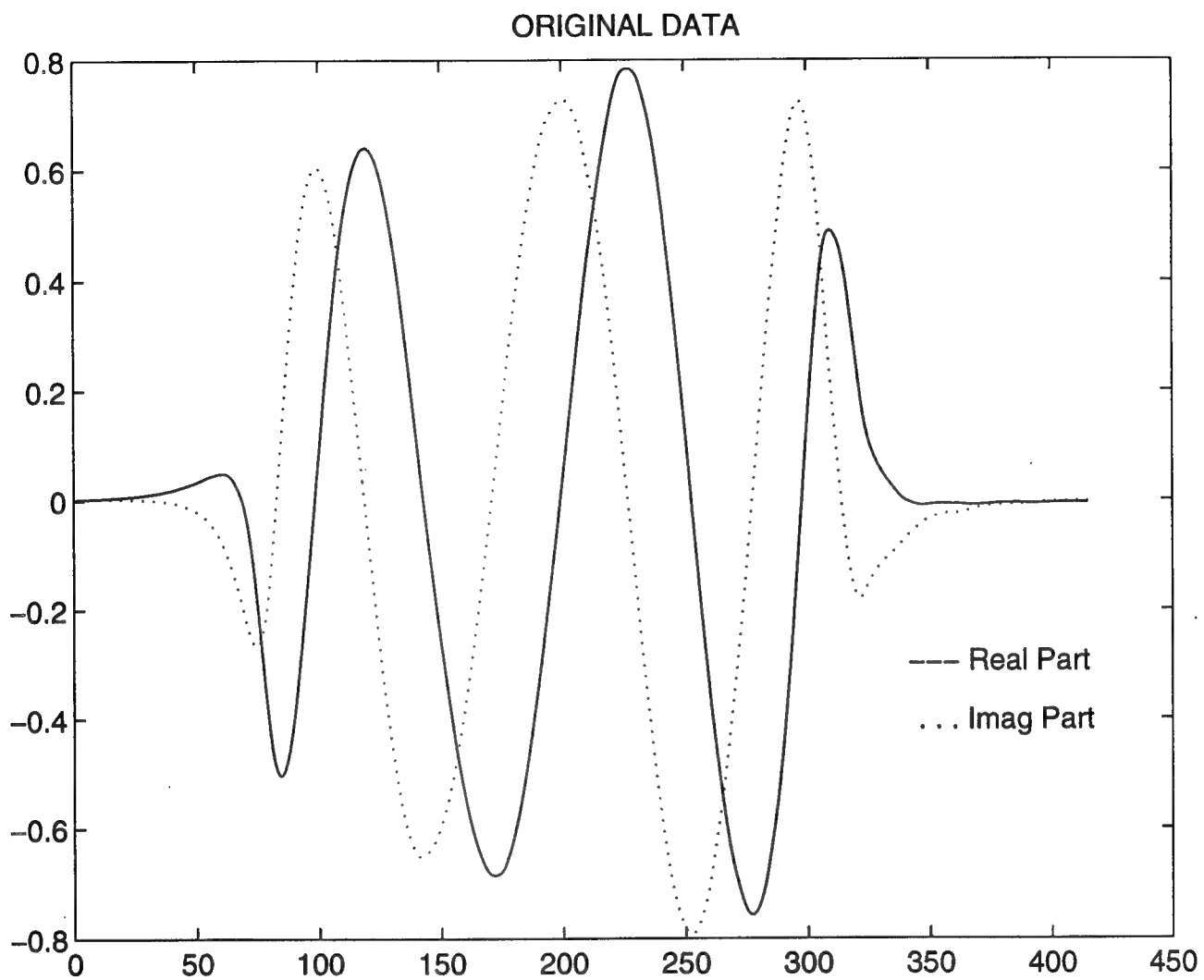
Both the direct methods utilizing analytic continuation and the iterative method using properties of the Hilbert Transform are very efficient tools for interpolation or extrapolation of frequency domain responses. The above examples demonstrate the implementation of the two different algorithms and compares the performance of each method.

## 7. REFERENCES

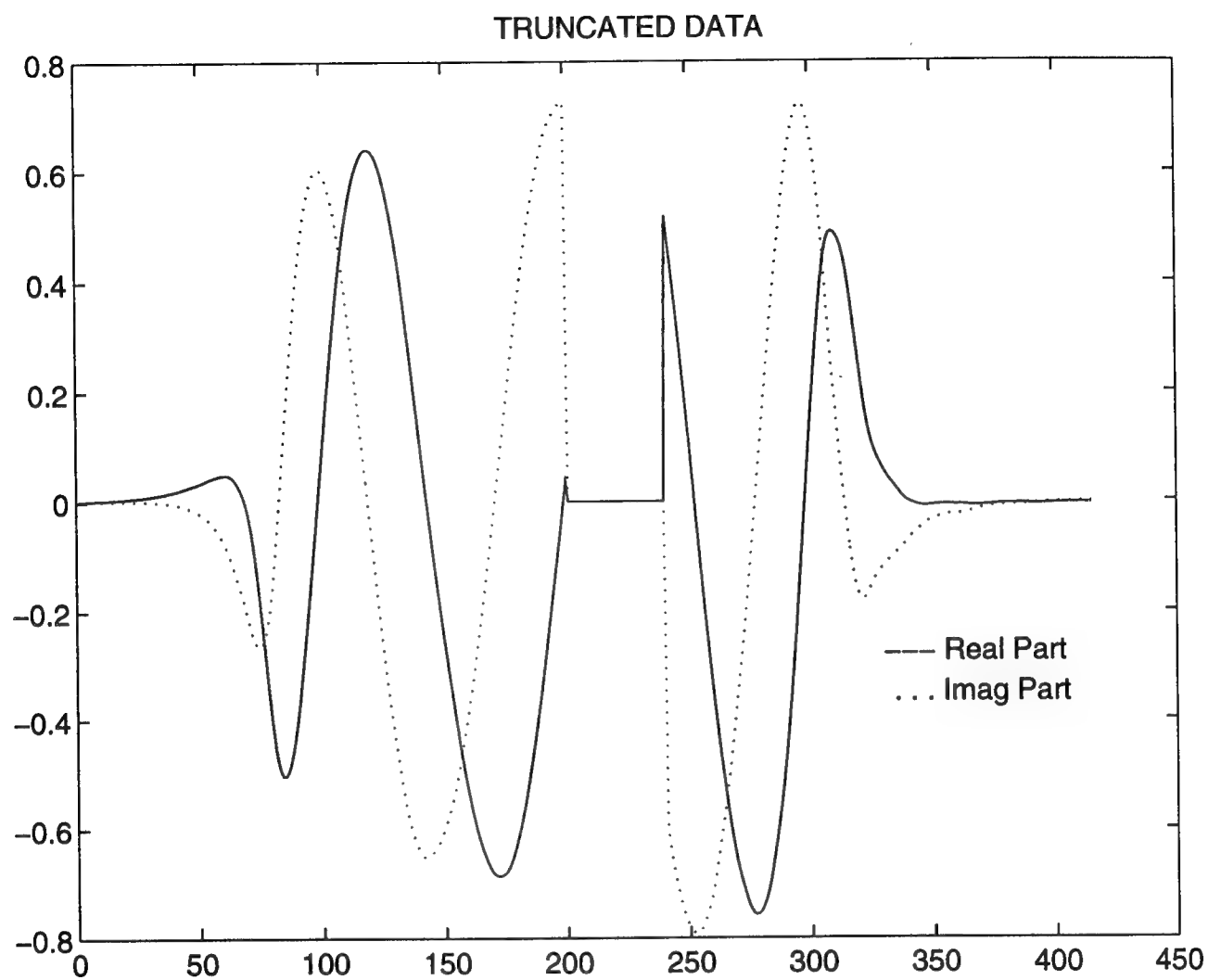
- [1] K. Kottapalli, T. K. Sarkar, Y. Hua, E. Miller and G. I. Burke, "Accurate Computation of Wide-Band Response of Electromagnetic Systems Utilizing Narrowband Information", IEEE Trans. on MTT, Vol. 39, pp. 682-688, April 1991.
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- [3] S. Van Huffel, "Analysis of the Total Least Squares Problem and its use in Parameter Estimations", Department Electrotechniek, Katholieke Universiteit Leuven, Ph.D. Dissertation.
- [4] A. V. Oppenheim and R. W. Schaffer, "Discrete-Time Signal Processing", Prentice Hall, Englewood Cliffs, NJ, 1989.

### Figure Captions

- 1(a) Original Data (415 points).
- 1(b) Data with 40 missing points.
- 1(c) Comparison of the reconstruction using the two methods (Real Part).
- 1(d) Comparison of the reconstruction using the two methods (Imaginary Part).
- 2(a) Data with 60 missing points.
- 2(b) Comparison of the reconstruction using the two methods (Real Part).
- 2(c) Comparison of the reconstruction using the two methods (Imaginary Part).
- 3(a) Data with 80 missing points and Non-zero initial guess.
- 3(b) Comparison of the reconstruction using the two methods (Real Part). Cutoff in Cauchy  $10^{-16}$ .
- 3(c) Comparison of the reconstruction using the two methods (Imaginary Part). Cutoff in Cauchy  $10^{-16}$ .
- 3(d) Comparison of the reconstruction using the two methods (Real Part). Cutoff in Cauchy  $10^{-21}$ .
- 3(e) Comparison of the reconstruction using the two methods (Imaginary Part). Cutoff in Cauchy  $10^{-21}$ .
- 4(a) Original Data (468 points).
- 4(b) Truncated Data with Initial Guess (40 missing points).
- 4(c) Comparison of the reconstruction using the two methods (Real Part).
- 5(a) Truncated Data with Initial Guess (60 missing points).
- 5(b) Comparison of the reconstruction using the two methods (Real Part).
- 5(c) Comparison of the reconstruction using the two methods (Imaginary Part).
- 6(a) Truncated Data with Initial Guess (60 missing points).
- 6(b) Comparison of the reconstruction using the two methods (Real Part).
- 6(c) Comparison of the reconstruction using the two methods (Imaginary Part).



**Figure 1(a)** Original Data (415 points)



**Figure 1(b)** Data with 40 missing points

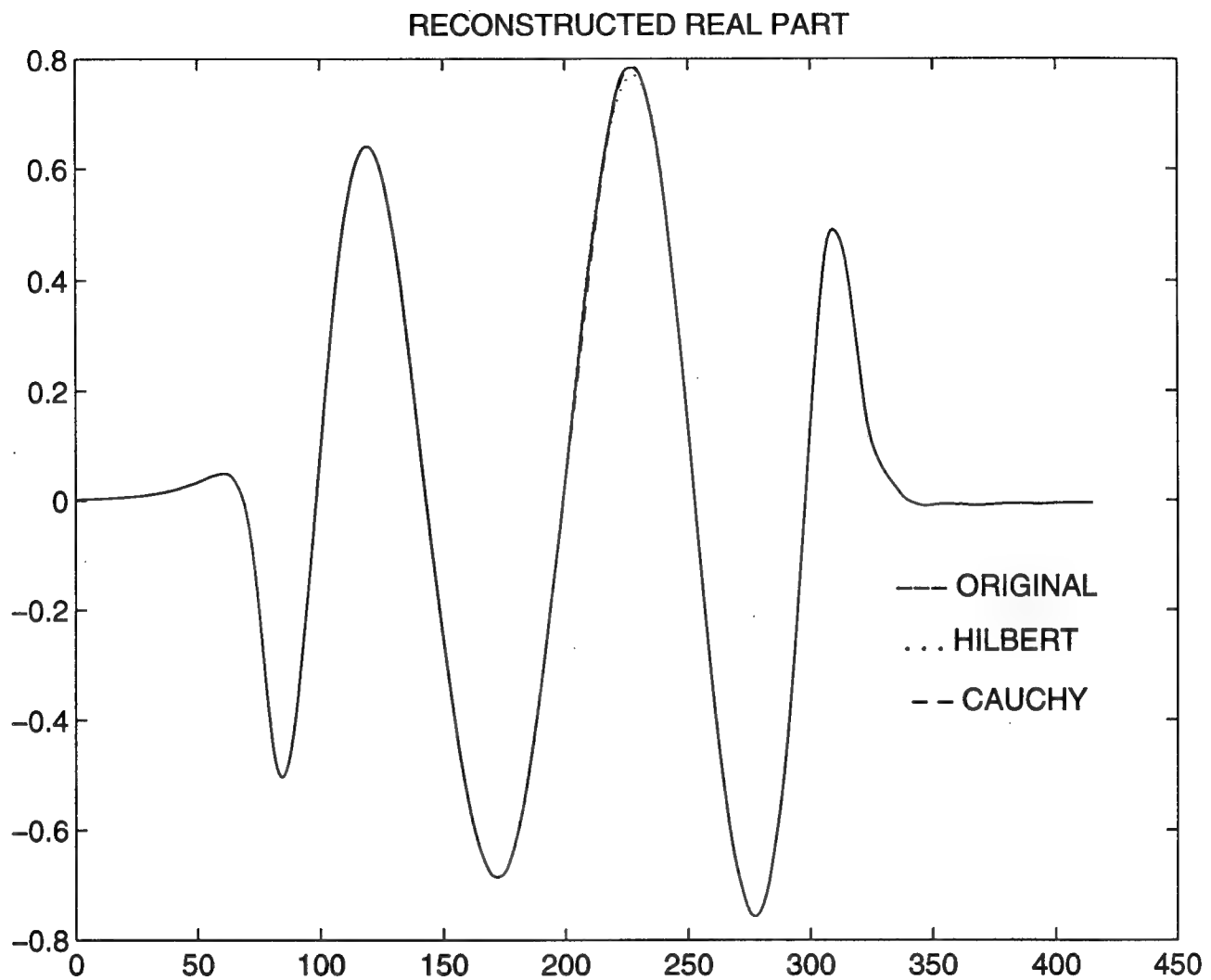


Figure 1(c) Comparision of the reconstruction using the two methods (Real Part)

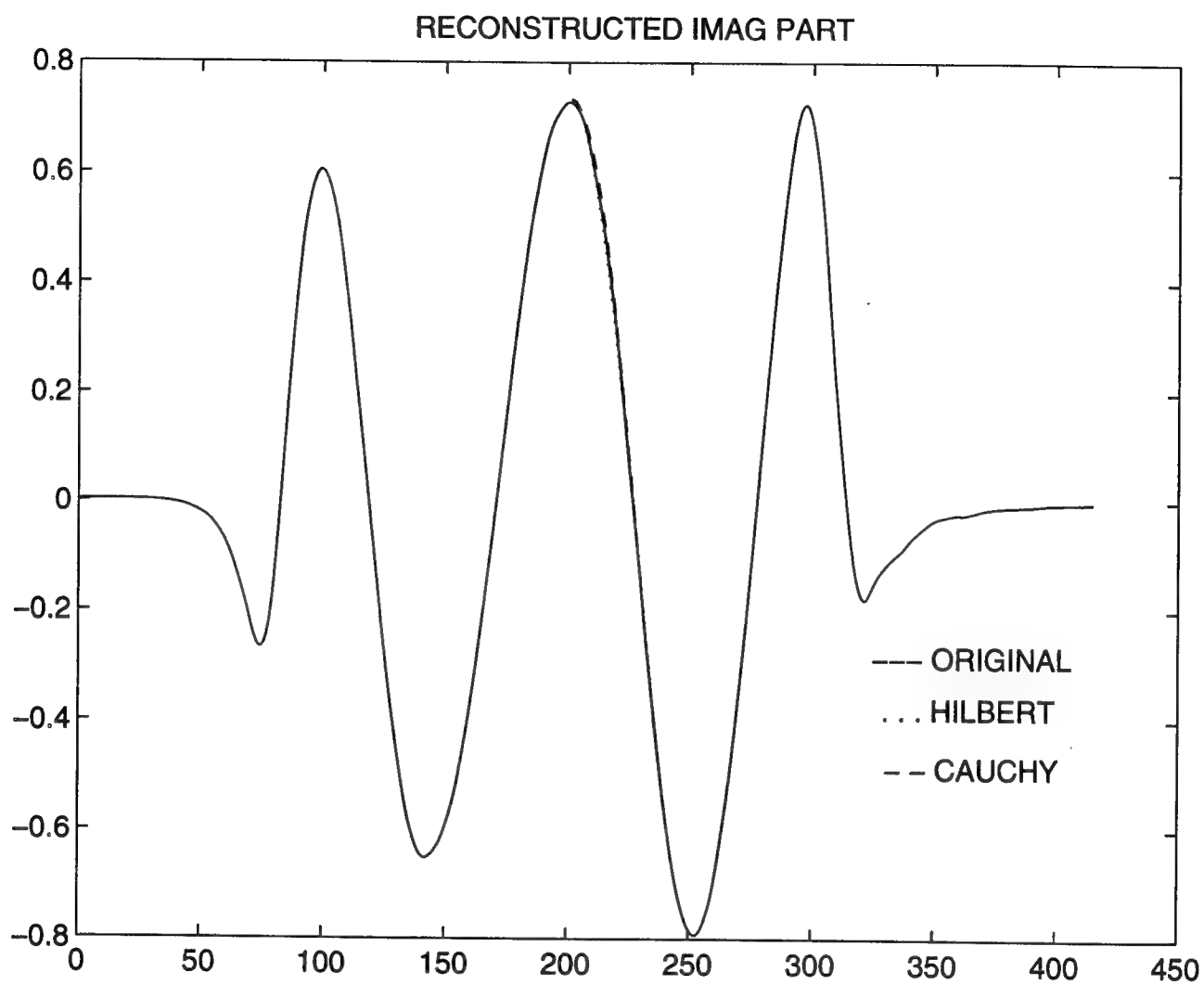


Figure 1(d) Comparison of the reconstruction using the two methods (Imaginary Part)

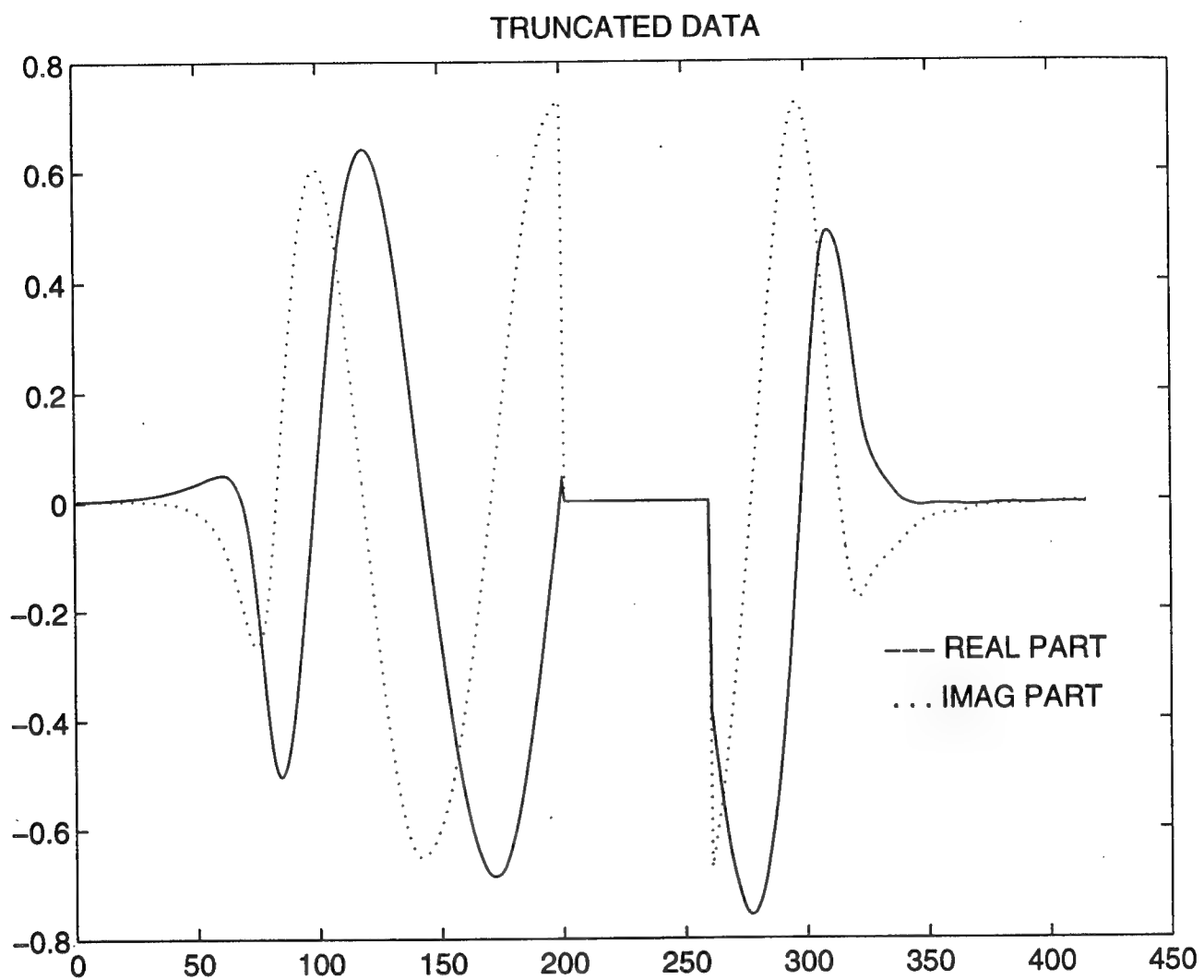


Figure 2(a) Data with 60 missing points

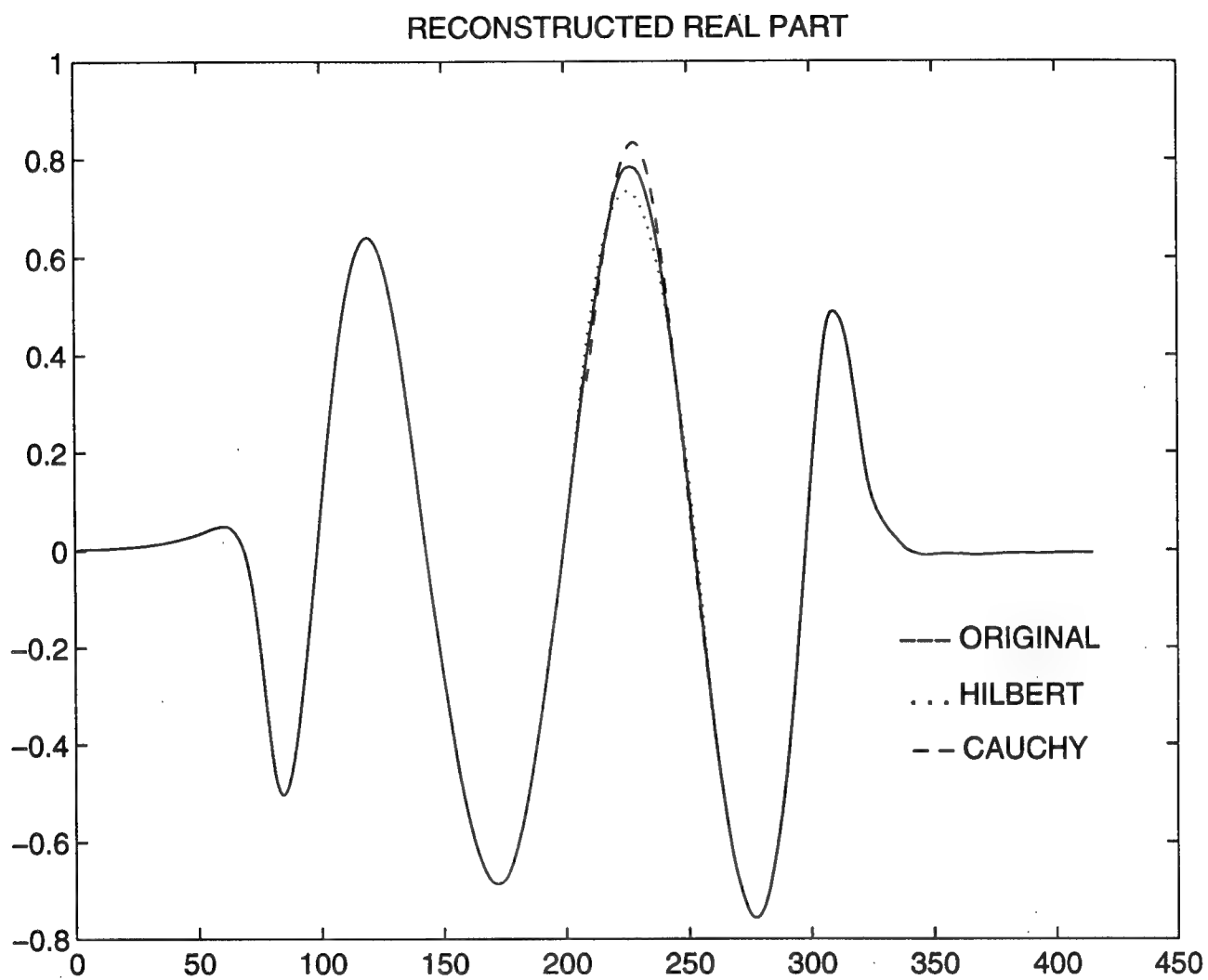


Figure 2(b) Comparison of the reconstruction using the two methods (Real Part)

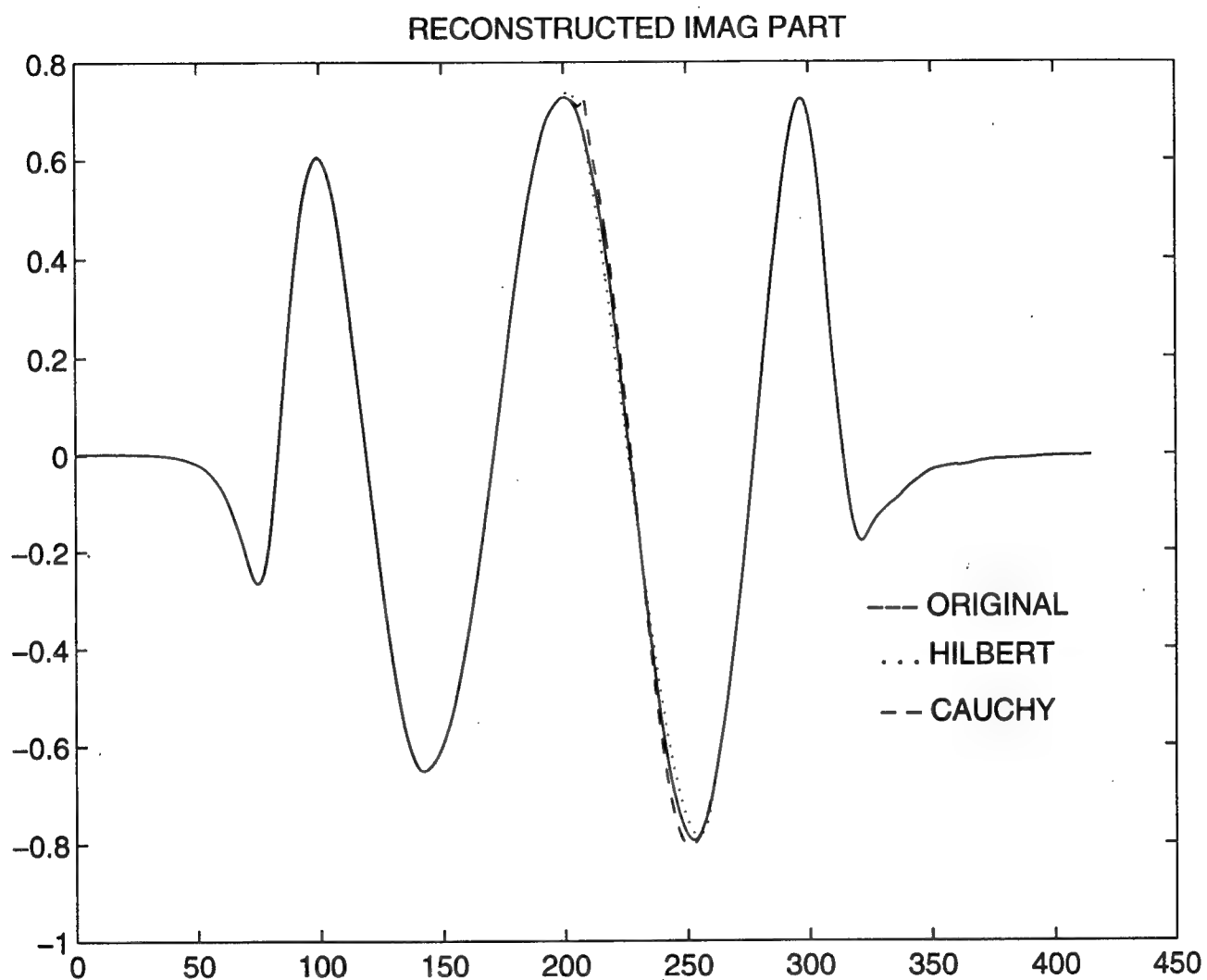


Figure 2(c) Comparision of the reconstruction using the two methods (Imaginary Part)

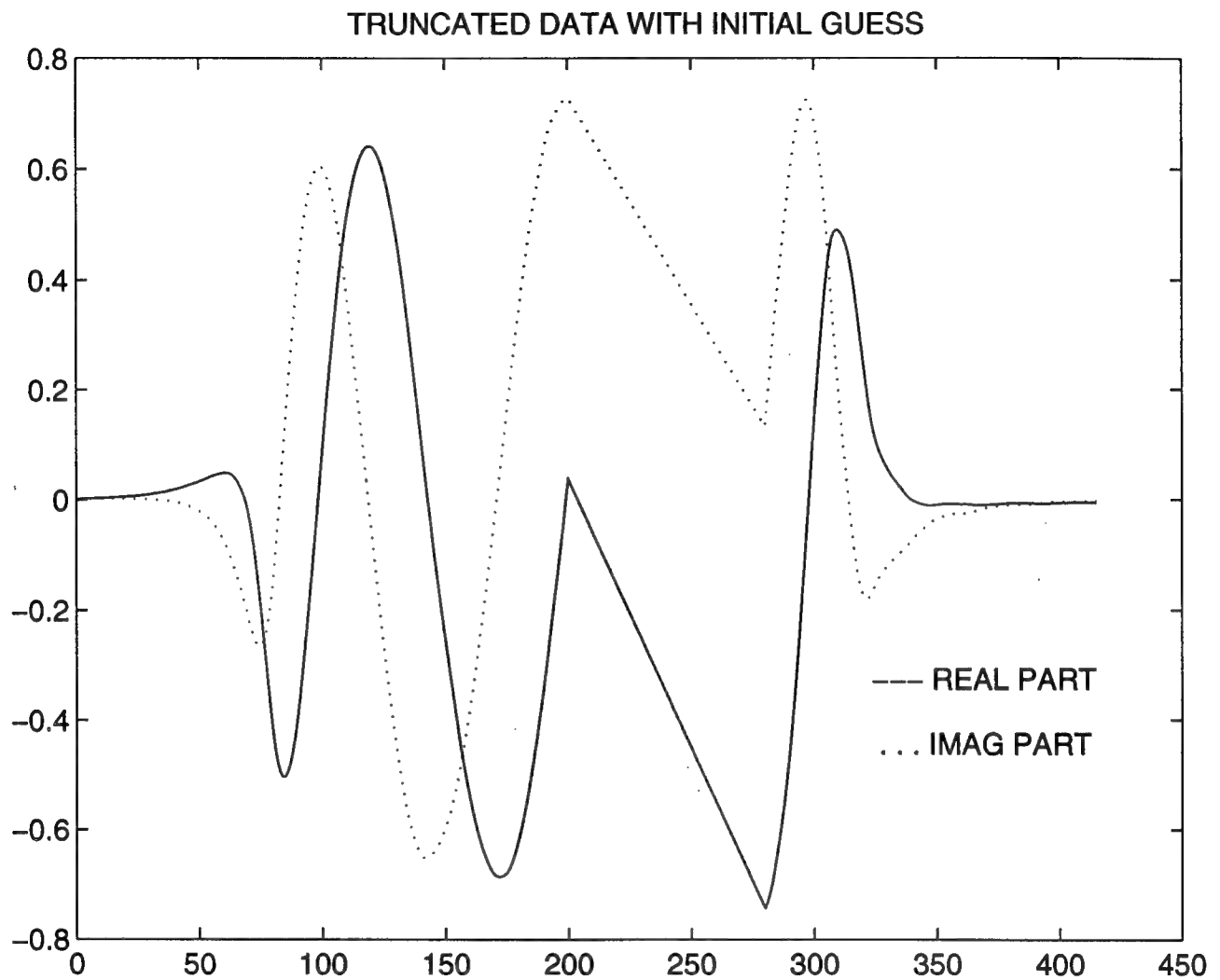
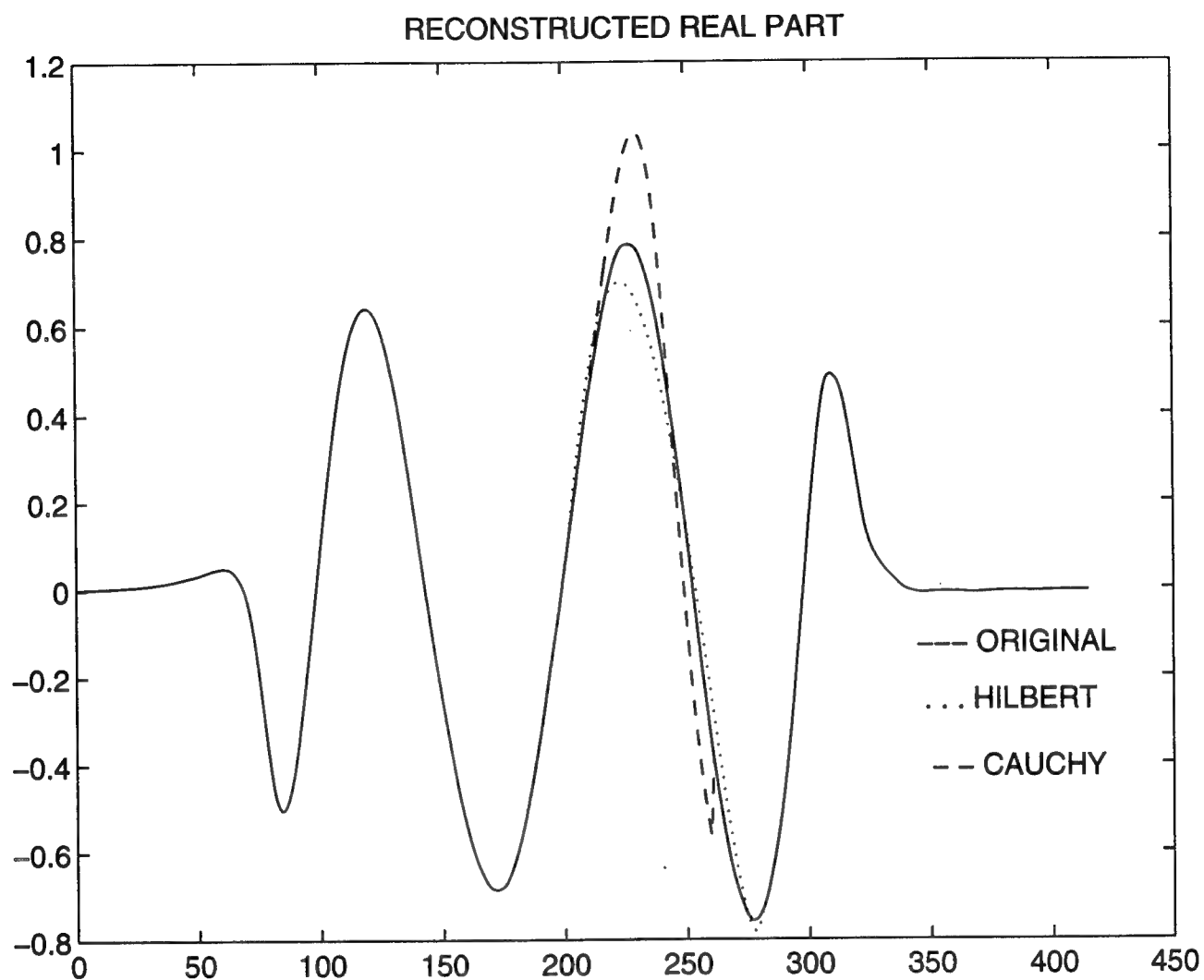
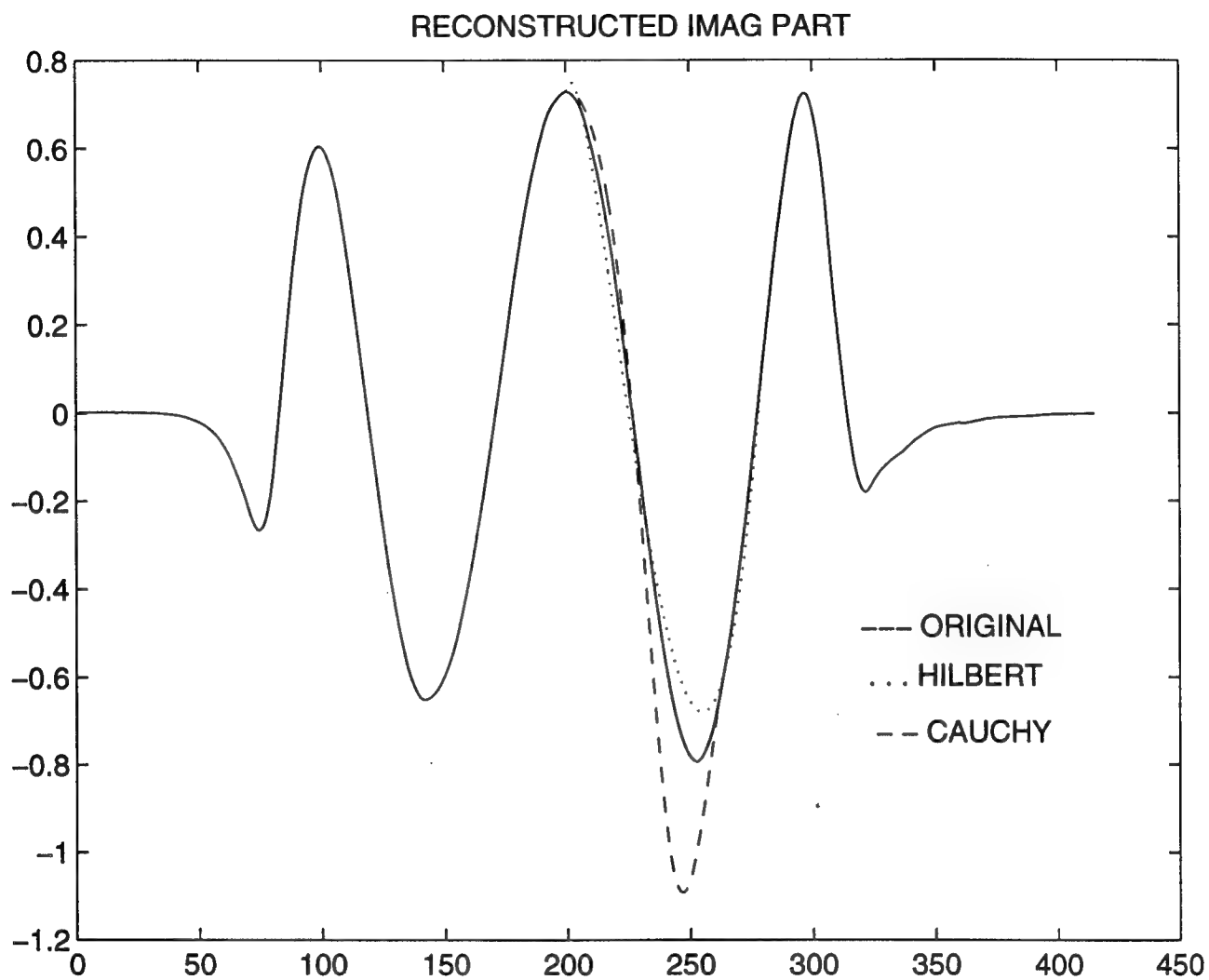


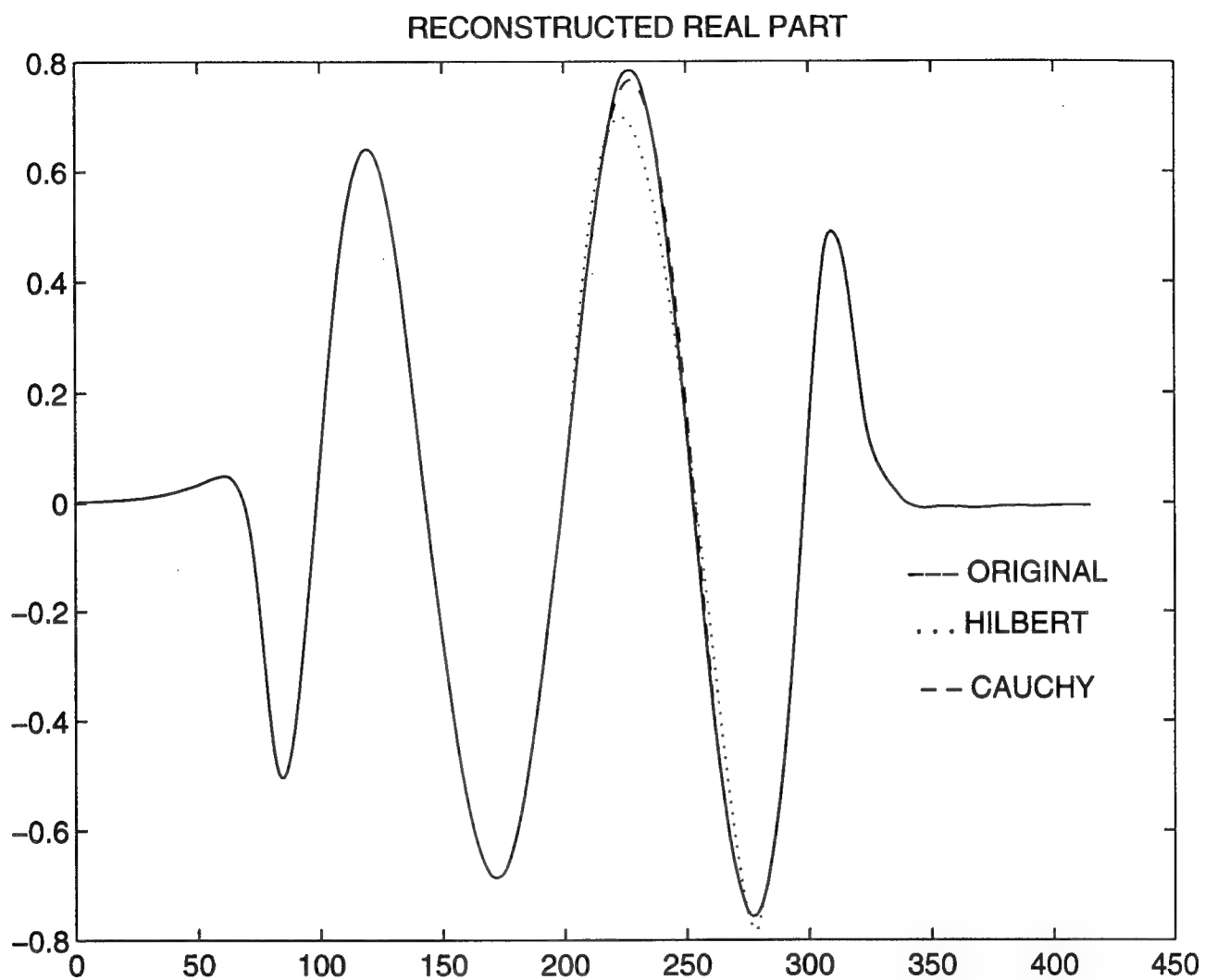
Figure 3(a) Data with 80 missing points and Non-zero initial guess.



**Figure 3(b)** Comparison of the reconstruction using the two methods (Real Part) Cutoff in Cauchy  $10^{-16}$



**Figure 3(c)** Comparison of the reconstruction using the two methods (Imaginary Part) Cutoff  
in Cauchy  $10^{-16}$



**Figure 3(d)** Comparison of the reconstruction using the two methods (Real Part) Cutoff in Cauchy  $10^{-21}$

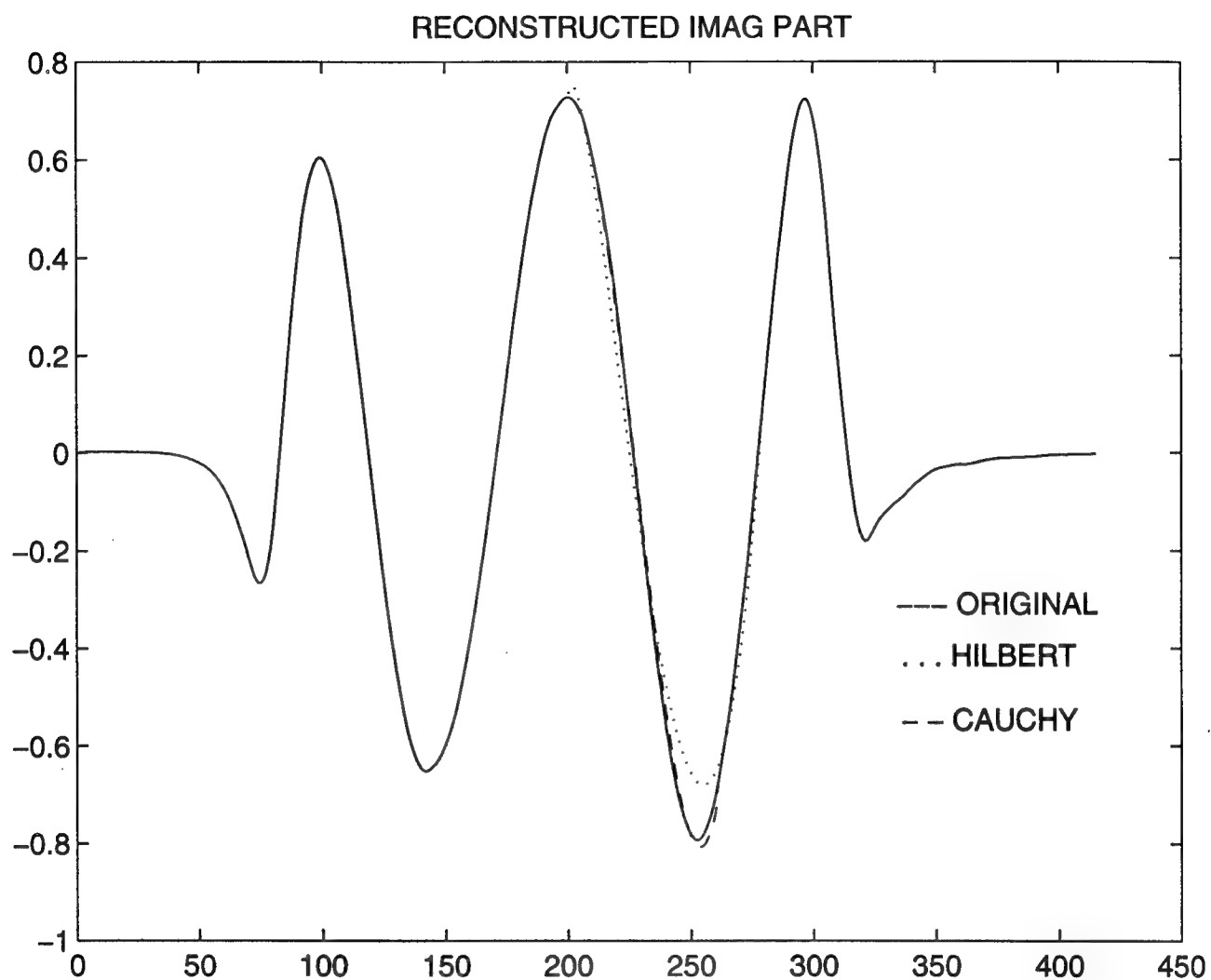


Figure 3(e) Comparison of the reconstruction using the two methods (Imaginary Part) Cutoff  
in Cauchy  $10^{-21}$

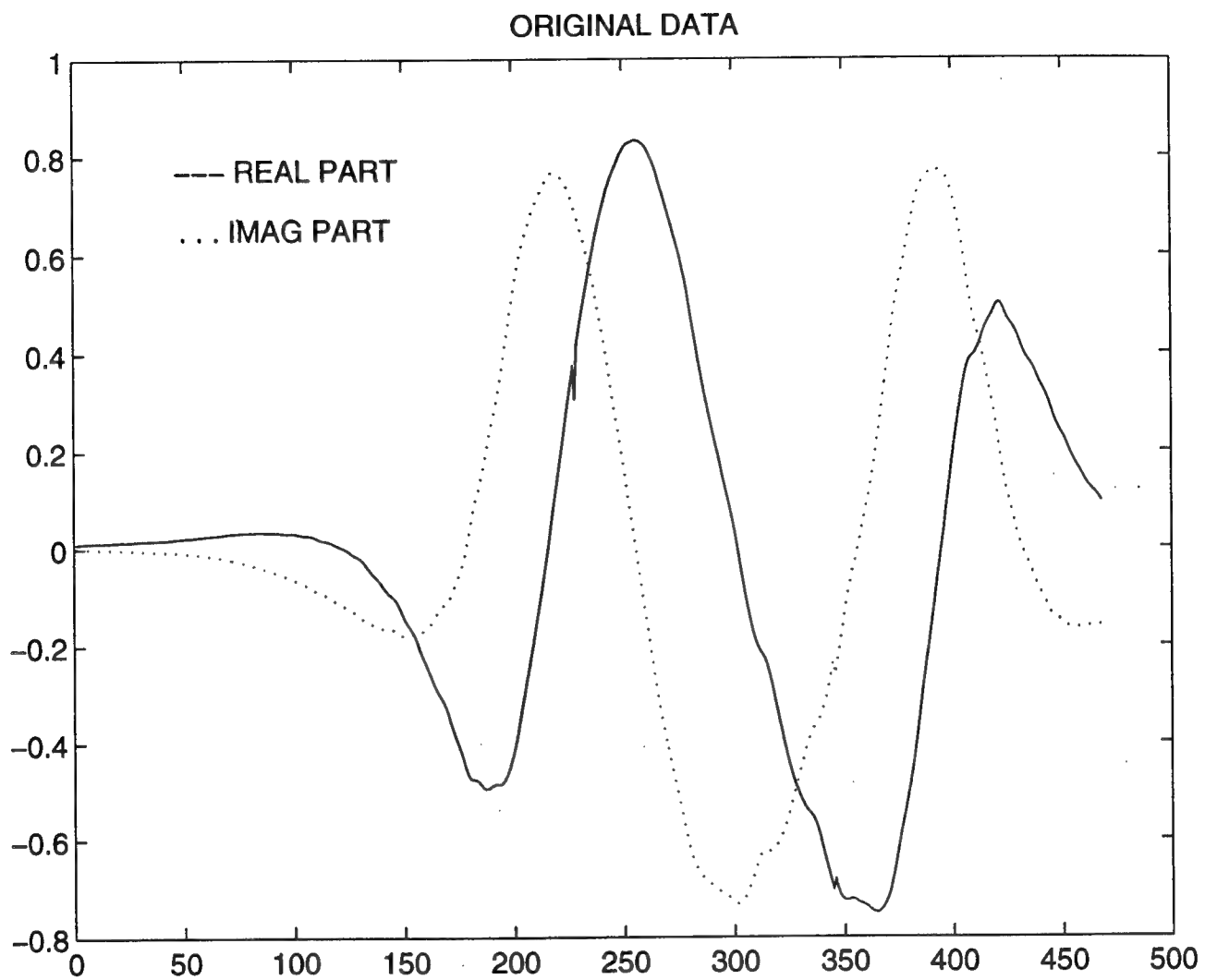


Figure 4(a) Original Data (468 points)

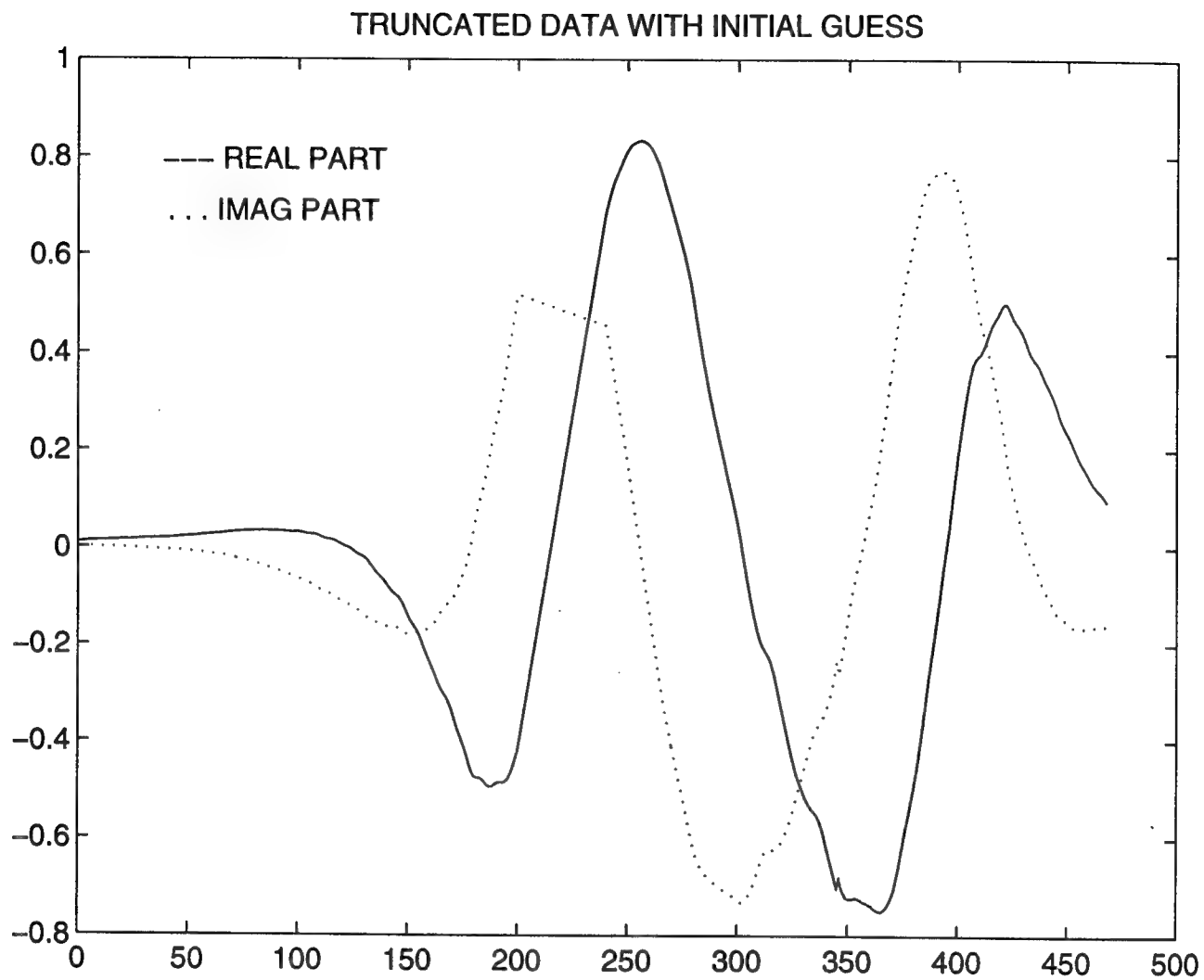


Figure 4(b) Truncated Data with Initial Guess (40 missing points)

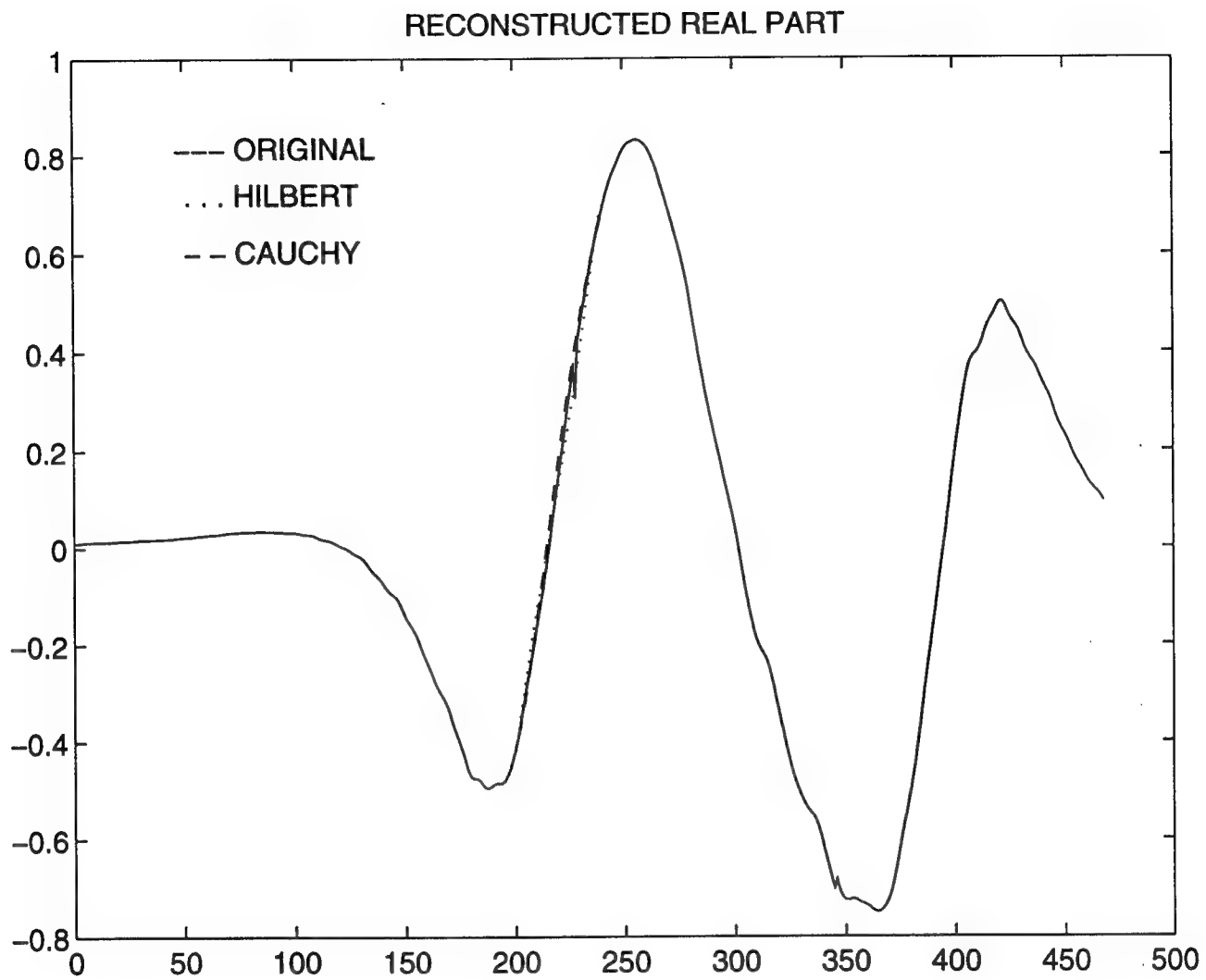


Figure 4(c) Comparison of the reconstruction using the two methods (Real Part)

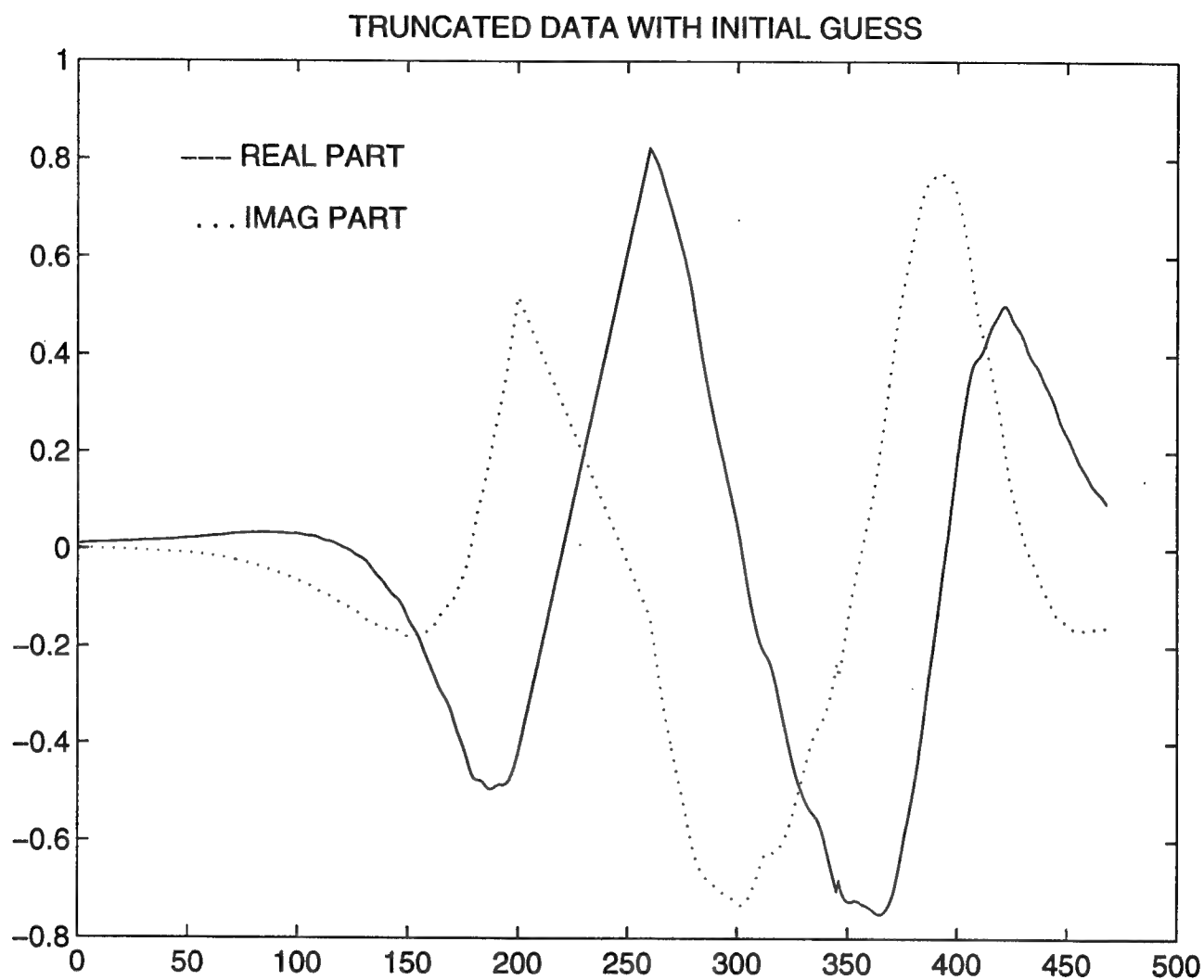
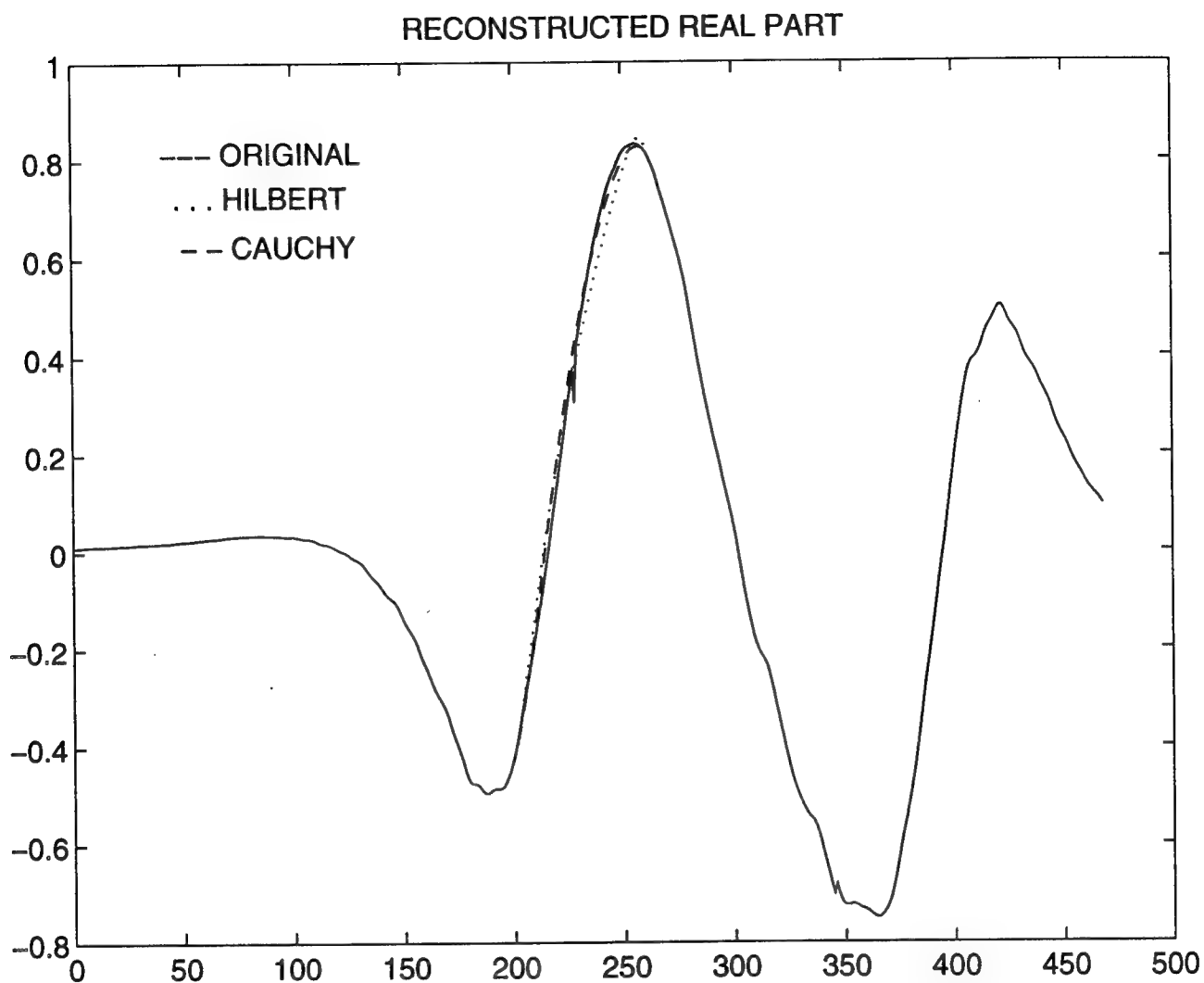


Figure 5(a) Truncated Data with Initial Guess (60 missing points)



**Figure 5(b)** Comparison of the reconstruction using the two methods (Real Part)

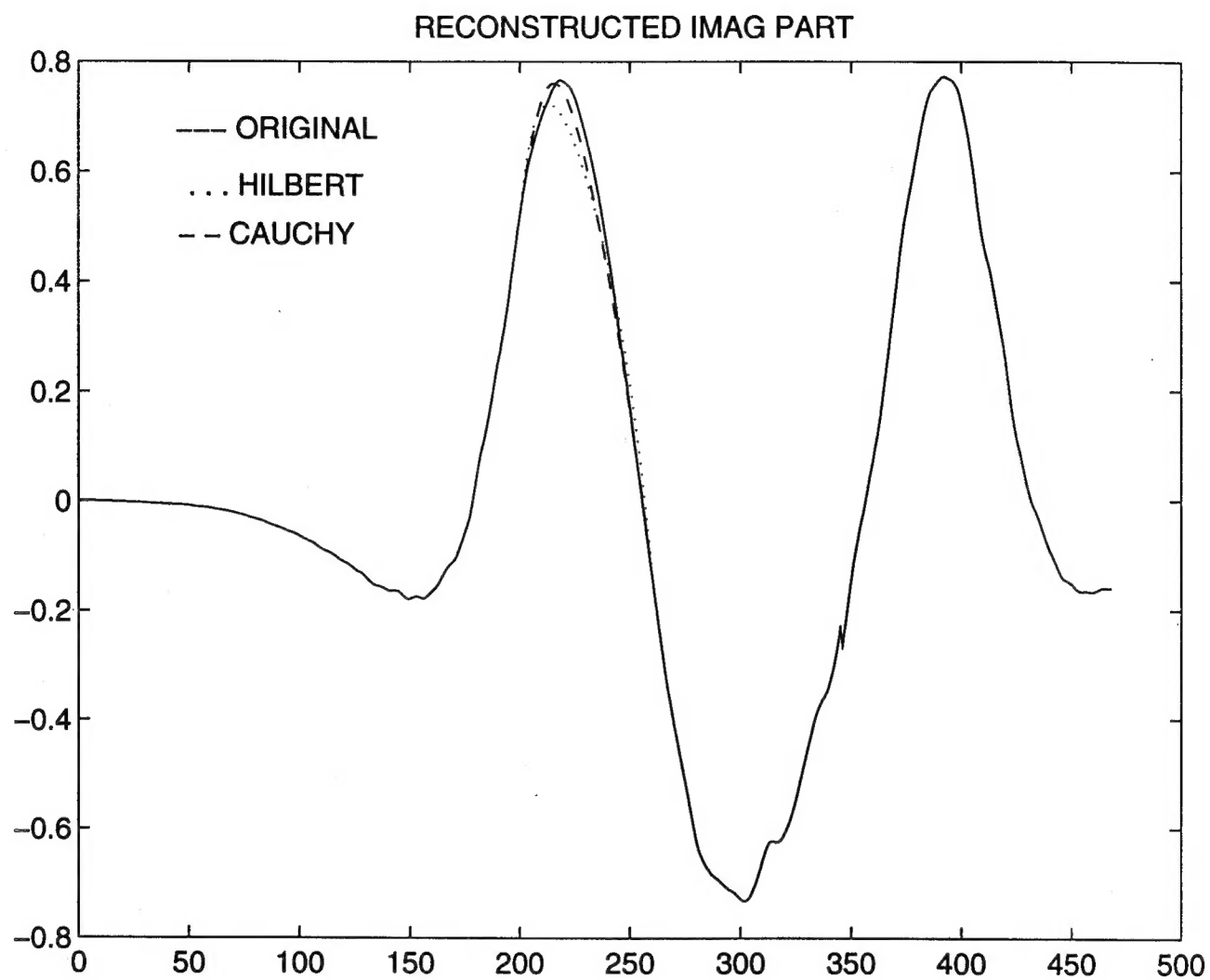


Figure 5(c) Comparison of the reconstruction using the two methods (Imaginary Part)

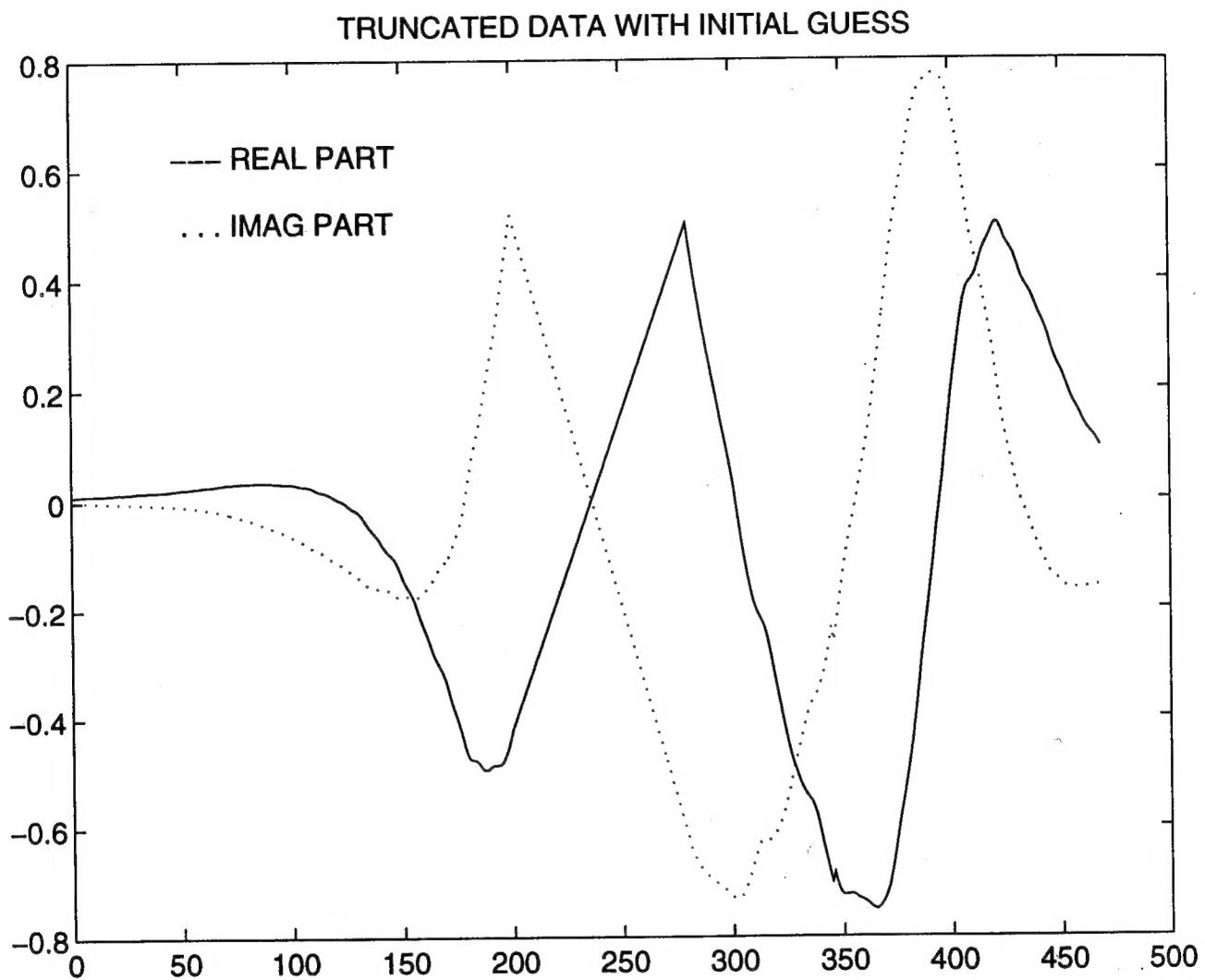


Figure 6(a) Truncated Data with Initial Guess (60 missing points)

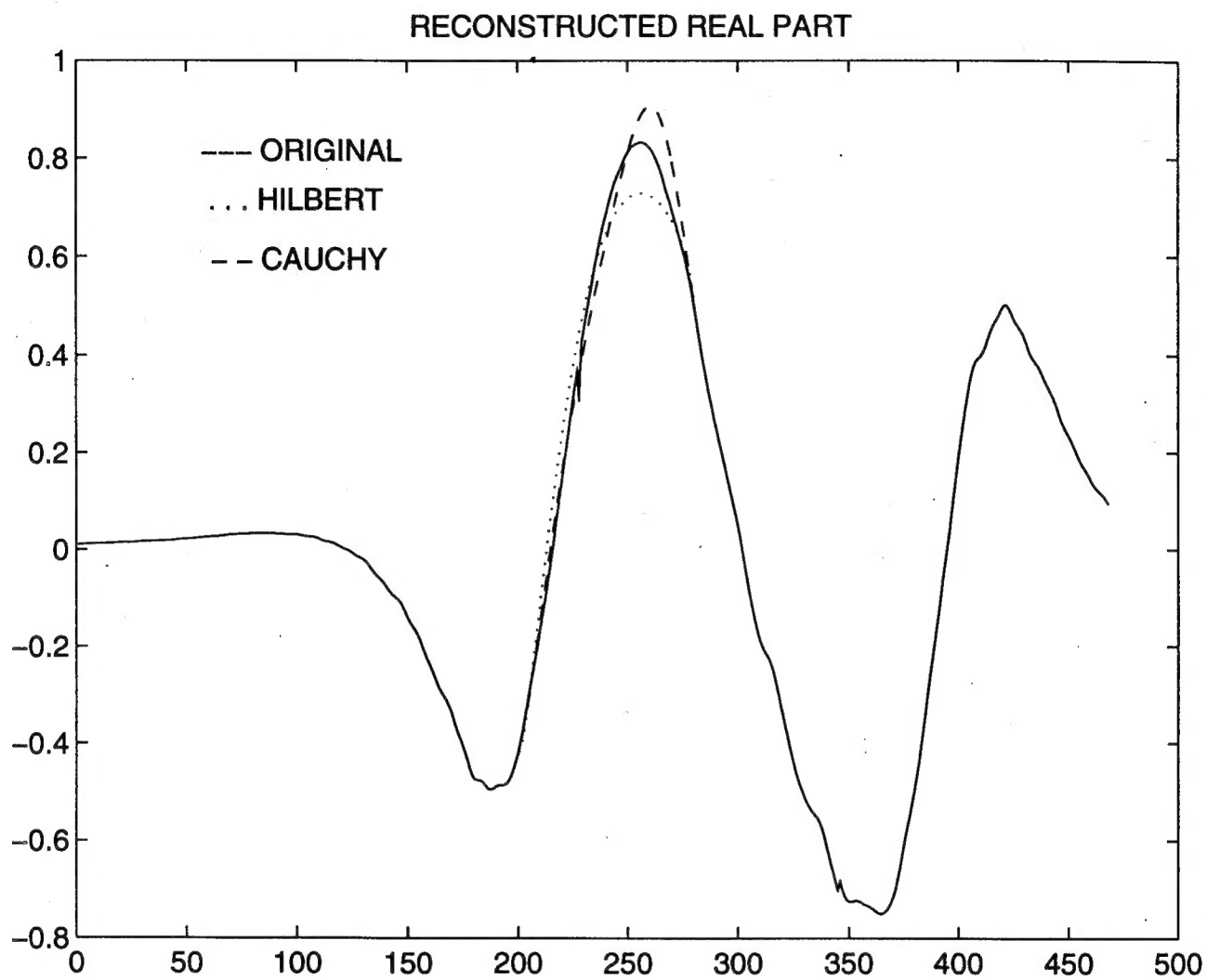
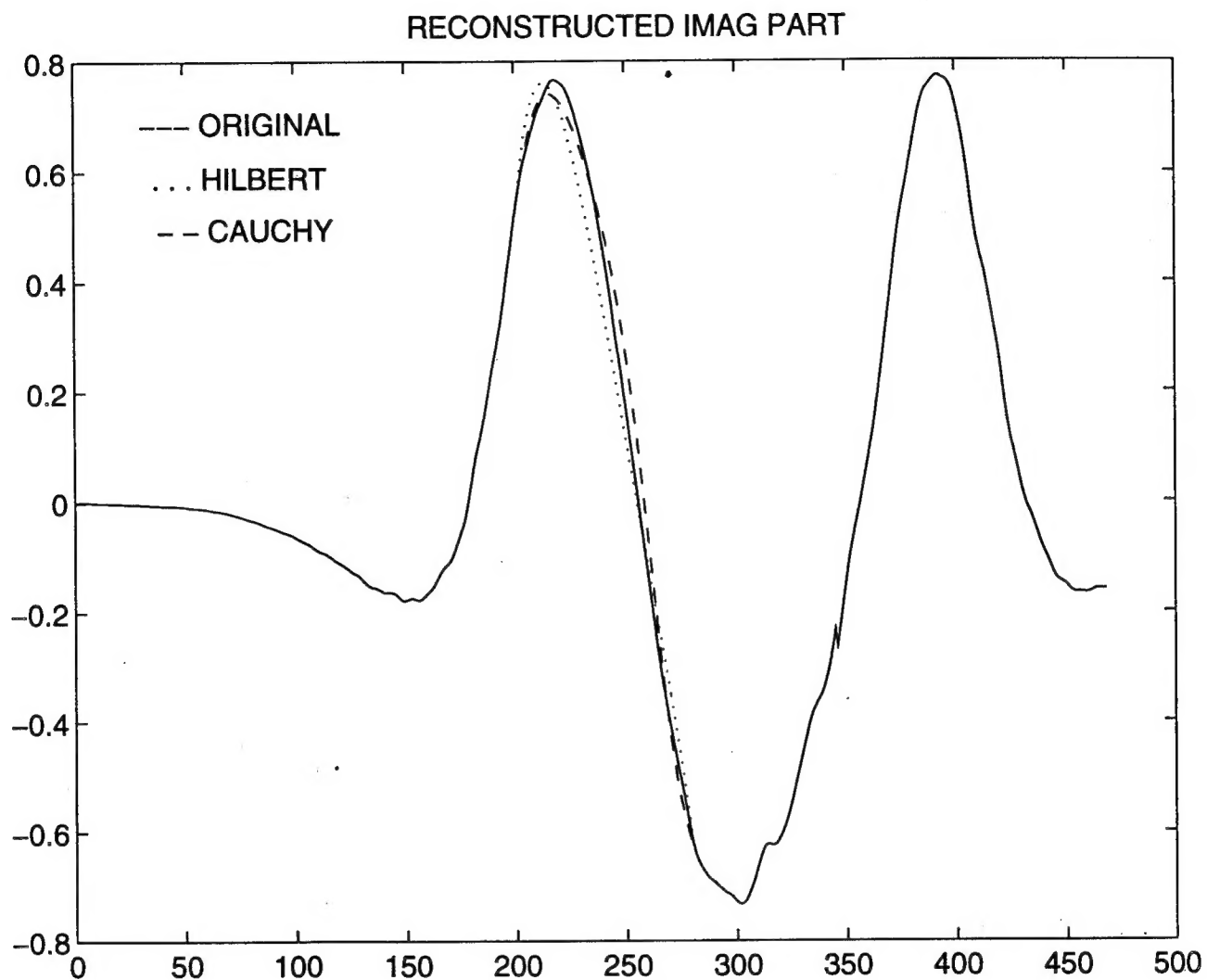


Figure 6(b) Comparison of the reconstruction using the two methods (Real Part)



**Figure 6(c)** Comparison of the reconstruction using the two methods (Imaginary Part)